MATROIDS AND GEOMETRIC INVARIANT THEORY OF TORUS ACTIONS ON FLAG SPACES

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ABSTRACT. Let $F//T$ be a G.I.T. quotient of a flag manifold F by the natural action of the maximal torus T in $\mathrm{SL}(n, \mathbb{C})$. The construction of the quotient space depends upon the choice of a T-linearized line bundle L of F. This note concerns the case where $L = L_\lambda$ is a very ample homogeneous line bundle determined by a dominant weight λ.

A theorem of Gel'fand, Goresky, MacPherson, and Serganova about matroids and matroid polytopes is applied to study semistability of flags relative to a given T-linearization of L_λ. The main theorem of this note is the follwing: if there exists a nonzero T-invariant global section of L_λ, then for each semistable flag $x \in F$, there exists a T-invariant global section s of L_λ such that $s(x) \neq 0$. Hence the global T-invariant sections of L_λ determine a well-defined map from $F//T$ to projective space, provided there is at least one such which is nonzero.

A related result in this note is that the closure of any T-orbit in F is projectively normal for any projective embedding of F. The proof of this fact uses essentially the same argument given for the semistability theorem above.

CONTENTS

1. INTRODUCTION

The geometry (both symplectic and algebraic) of the quotients $F//T$ have been extensively studied in recent years; Allen Knutson called them "weight

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varieties"[1] in his thesis [K]. The dependence of the geometry of the quotient on the choice of linearization was studied by Yi Hu [Hu] and the cohomology of weight varieties was computed by Rebecca Goldin [Go]. Special cases of weight varieties have been studied since the nineteenth century; for example a G.I.T. quotient $(\mathbb{CP}^{k-1})^n//\mathrm{PGL}(k,\mathbb{C})$ is isomorphic to a G.I.T. quotient $\mathrm{Gr}_k(\mathbb{C}^n)//T$ by the Gel'fand MacPherson correspondence (here $\mathrm{Gr}_k(\mathbb{C}^n)$ denotes the Grassmannian). The projective invariants of n-tuples of points on projective space are still not understood today; we do not even know a minimal set of generators for the ring of projective invariants (see page 8 of [Ha]).

We take one step towards a solution to the generators problem (for $G = \mathrm{SL}(n,\mathbb{C})$) with Theorem 2.2, which implies that the lowest degree T–invariants in the graded ring of F are sufficient to give a well-defined map from $F//T$ to projective space. Consequently these global sections determine an ample line bundle M of $F//T$. We are left with the problem of determining which tensor power of M is very ample.

The proof of Theorem 2.2 involves a simple combinatorial argument involving Minkowski sums of weight polytopes of flags. These weight polytopes are also known as flag matroid polytopes, see [BGW]. Two facts are essential to the argument:

- Any subset of $\mathrm{SL}(n,\mathbb{C})$ roots which are linearly independent may be extended to a basis of the root lattice.
- Each edge of a matroid polytope is parallel to a root of $\mathrm{SL}(n,\mathbb{C})$ (due to Gel'fand, Goresky, MacPherson, Serganova).

Remark 1.1. *The first fact is specific to* $\mathrm{SL}(n,\mathbb{C})$. *The root systems of other classical complex simple Lie algebras do not have this remarkable saturation property. However, the second result is a special case of the Gel'fand–Serganova theorem which is one of the central theorems in the new subject of Coxeter matroids, see* [BGW]. *It should be noted that Theorem 2.2 easily follows from a theorem of Neil White* [W] *in the case that F is a Grassmannian.*

Additionally, the tools we develop in proving Theorem 2.2 also allow us to show that the closure of a T-orbit $cl(T\cdot x)$ for any $x \in F$ (for any projective embedding of F) is a projectively normal toric variety. Again Neil White [W] showed this holds when F is a Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$. Additionally, R. Dabrowski [Dab] proved that projective normality holds for closures of certain *generic* T-orbits in other homogeneous spaces G/P (he covered the case that G is any semi-simple complex Lie group).

[1]The term "weight variety" actually refers to more general quotients; they are G.I.T. quotients of G/P by a maximal torus T in G, where G is a reductive connected complex Lie group and P is a parabolic subgroup of G containing T.

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2. The construction of $F//T$ and statement of main theorem

A weight variety of $G = \mathrm{SL}(n, \mathbb{C})$ is a G.I.T. quotient of a flag manifold $F = G/P$ by the action of the Cartan subgroup T. The construction of such a quotient involves the choice of a T-linearized line bundle L of $F = G/P$. If L is very ample, then it's isomorphism class is determined by a choice of dominant weight λ such that P is the largest parabolic subgroup such that the character e^{λ} defined on the Borel subgroup B of upper triangular matrices extends (uniquely) to P. The T-linearization of L will also depend on a choice of a weight μ, but μ need not be dominant.

2.1. Elementary notions from the representation theory of $\mathrm{SL}(n, \mathbb{C})$.

Since $\mathrm{SL}(n, \mathbb{C})$ is simply connected, the set of $\mathrm{SL}(n, \mathbb{C})$ weights are the differentials evaluated at the identity element of characters $\chi : T \to \mathbb{C}^*$, which are holomorphic homomorphisms (that is, the character lattice coincides with the weight lattice). The differential $d\chi$ (evaluated at the identity element of T) of χ lies within the dual Lie algebra $\mathfrak{t}^*$ of T. On the other hand, if $\varpi \in \mathfrak{t}^*$ is a weight, we shall denote e^{ϖ} as the unique character $e^{\varpi} : T \to \mathbb{C}^*$ such that $d(e^{\varpi}) = \varpi$.

A character e^{λ} applied to $t = \mathrm{diag}(t_1, \ldots, t_n) \in T$ must be equal to $\prod_{i=1}^{n} t_i^{a_i}$ for some fixed integers a_i. Since $\prod_{i=1}^{n} t_i = 1$ for all $t \in T$, we have that the n-tuple of exponents $(a_1, \ldots, a_n)$ and $(a_1 + a, a_2 + a, \ldots, a_n + a)$ determine the same character. We may thus view the abelian group of characters of T as $\mathbb{Z}^n/\Delta$ where Δ is the diagonal. On the other hand, the weight $\lambda \in \mathfrak{t}^*$ takes a complex vector $(z_1, \ldots, z_n) \in \mathfrak{t}$ (where $z_1 + z_2 + \cdots + z_n = 0$) to $\sum_{i=1}^{n} a_i z_i$. Again, adding a constant to each a_i results in the same function, and so again we have that the additive group of weights is isomorphic to $\mathbb{Z}^n/\Delta$. We shall henceforth identify characters and weights as n-tuple of integers modulo the diagonal Δ.

2.1.1. *Dominant weights and construction of very ample line bundles.* We say that a weight $\lambda = (a_1, \ldots, a_n)$ is dominant if the a_i's are non-strictly decreasing. Now suppose that λ is dominant, and $P \subset G$ is the largest parabolic subgroup (a subgroup containing all the upper triangular matrices in G) such that e^{λ} extends to a character $\chi : P \to \mathbb{C}^*$. It is a basic fact that χ is determined by it's restriction e^{λ} to the torus T, so we will abuse notation and identify χ with e^{λ}.

The dominant weight λ determines a very ample line bundle L_{λ} of G/P. The total space of L_{λ} is the set of equivalence classes of pairs (g, z) for $g \in G$ and $z \in \mathbb{C}$, where $(g, z) \sim (gp, e^{\lambda}(p)z)$ for all $p \in P$. The map π from the total space to G/P is given by $\pi : (g, z) \mapsto gP$. Each global section of L_{λ}

is given by $s_f(gP) = (g, f(g))$ where $f : G \to \mathbb{C}$ is a holomorphic function such that $f(gp) = e^{\lambda}(p)f(g)$ for all $p \in P$ and $g \in G$.

There is a natural action of G on the total space of L_λ, given by $g \cdot (g', z) = (gg', z)$. This defines an action on sections by

$$(g \cdot s)(g'P) = g \cdot s(g^{-1}g'P) = g \cdot (g^{-1}g', f(g^{-1}g')) = (g'P, f(g^{-1}g')).$$

The vector space V_λ of global sections is an irreducible representation of G; the action of $g \in G$ on s_f is $(g \cdot s_f)(g'P) = g \cdot s_f(g^{-1}g'P)$.

The N-th tensor power $L_\lambda^{\otimes N}$ is isomorphic to $L_{N\lambda}$.

2.1.2. *Choosing a T-linearization of L_λ.* There is a canonical T-linearization of L_λ, given by restricting the action of G on L_λ to T. We shall call this the "democratic" linearization. A weight μ may be used to twist the democratic linearization;

$$t \cdot (g, z) = (tg, \mu(t)z).$$

We shall call this the "μ–linearization". Indeed the set of all T-linearizations are given by the characters μ of T.

The μ-twisted action of T on a section s_f is given by the formula,

$$(t \cdot s_f)(gP) = (gP, e^{\mu}(t)f(t^{-1}g).$$

Hence s_f is T-invariant iff $\mu(t)f(t^{-1}g) = f(g)$ for all $t \in T$. Equivalently, we have that s_f is T-invariant iff for all $t \in T$ and $g \in G$,

$$f(tg) = e^{\mu}(t)f(g).$$

The action on a section s_f of $L_\lambda^{\otimes N}$ is given by $(t \cdot s_f)(gP) = (gP, e^{N\mu}(t)f(g))$, and so the T-invariant sections s_f of the N-th tensor power of L_λ are those which satisfy

$$f(t \cdot g) = e^{N\mu}(t)f(g).$$

2.2. **The G.I.T. construction.** The G.I.T. quotient $F//T$ associated to the pair (λ, μ) is the projective variety,

$$F//T = \mathrm{Proj}\left(\bigoplus_{N=0}^{\infty} \Gamma(F, L_\lambda^{\otimes N})^T\right),$$

where T acts on L_λ via the μ-linearization.

Definition 2.1. *The set of semistable points $F^{ss} \subset F$ is defined by $p \in F^{ss}$ iff there exists some positive integer N and a T-invariant global section s of $L_\lambda^{\otimes N}$ such that $s(p) \neq 0$. (Normally there is the additional requirement that $X_s = \{p \in F \mid s(p) \neq 0\}$ is affine but this is automatic since F is a projective variety.) If we take the μ-linearization of L_λ, then we shall say that a semistable point is μ-semistable.*

A standard result of Mumford's Geometric Invariant Theory is that the G.I.T. quotient is topologically given as a quotient space of the open subset of semistable points. In particular there is a surjective continuous map $\pi : F^{ss} \to F//T$, where $\pi(x) = \pi(y)$ iff the closures of the T-orbits of x and

y (Zariski closure) have non-empty intersection in F^{ss}; $\overline{T \cdot x} \cap \overline{T \cdot y} \cap F^{ss} \neq \emptyset$. The space $F//T$ has the quotient topology relative to the surjective map π.

The proof of the following theorem will be given in section §4. This theorem allows us to explicitly construct an ample line bundle of $F//T$, and to cover $F//T$ by explicit affine varieties.

Theorem 2.2. *(Main Theorem) Suppose that λ is a dominant weight and μ is any weight, such that $\lambda - \mu$ lies in the root lattice of* $\mathrm{SL}(n, \mathbb{C})$*. Then if $p \in F$ is μ–semistable there is a global T-invariant section s of L_λ such that $s(p) \neq 0$.*

Remark 2.3. *If $\lambda - \mu$ is not in the root lattice, then $\Gamma(F, L_\lambda)^T$ is zero. In fact, $\Gamma(F, L_\lambda)^T$ is nonzero if and only if $\lambda - \mu$ is in the root lattice and μ lies within the convex hull of the Weyl orbit of λ. If μ does not lie in the convex hull of the Weyl orbit of λ, then $\Gamma(F, L_\lambda^{\otimes N})^T$ is zero for all $N > 0$; in this case there are no semistable points in F, and the quotient $F//T$ is empty.*

The following is taken from [Do], chapter 8. Let $s_1, \ldots, s_m$ be a basis of the T-invariant sections of L_λ for the μ-linearization. By theorem 2.2, the semistable points F^{ss} are covered by the affine open subsets X_{s_i}, where $X_{s_i} = \{x \in F \mid s_i(x) \neq 0\}$. Let Y_i be the affine quotient $X_{s_i}//T$; the affine coordinate ring of Y_i is $\mathcal{O}(X_{s_i})^T$. The Y_i's may be glued together via the transition functions s_i/s_j to form the G.I.T. quotient $F//T$, and simultaneously an ample line bundle M of $F//T$, such that $\pi^*(M) = \iota^*(L_\lambda)$, where $\iota : F^{ss} \to F$ is the inclusion map.

As stated in the introduction, it remains an open problem to compute the minimal integer N such that $M^{\otimes N}$ is very ample.

3. Matroid polytopes and weight polytopes

A matroid is a pair $M = (E, \mathcal{B})$ where E is a finite set called the ground set of M, and $\mathcal{B}$ is a nonempty collection of subsets of E called bases that satisfy the exchange condition, which is that for any two bases $B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \backslash B_2$ then there is an element $y \in B_2 \backslash B_1$ such that $(B_1 \backslash \{x\}) \cup \{y\} \in \mathcal{B}$ is a basis. Necessarily it follows that all bases $B \in \mathcal{B}$ have the same cardinality, which is called the rank of M. Matroids are a generalization of finite configurations of vectors, where the only data known about the set of vectors is which subsets are maximal independent subsets. The collection of maximal independent subsets satisfies the exchange condition. Similarly, a linear subspace Λ of dimension k of $\mathbb{C}^n$ determines a matroid $M(\Lambda)$, given by the vector configuration $\{\pi_\Lambda(e_1), \ldots, \pi_\Lambda(e_n)\}$ where π_Λ is orthogonal projection onto Λ (for the standard Hermitian form) and the e_i's are the standard basis vectors of $\mathbb{C}^n$.

3.1. **Matroid polytopes.** Suppose that $M = (E, \mathcal{B})$ is a matroid, and $E = \{1, 2, 3, \ldots, n\}$. For each $B \in \mathcal{B}$ let $v^B \in \mathbb{R}^n/\Delta$ (Δ is the diagonal in $\mathbb{R}^n$) be given by $v_i^B = 0$ if $i \notin B$ and $v_i^B = 1$ if $i \in B$. Let P_M be the convex

hull of $\{v^B \mid B \in \mathcal{B}\}$. We call P_M a matroid polytope. Each v^B is a vertex of P_M and so M may be recovered from P_M.

Theorem 3.1. *(Gel'fand Goresky MacPherson Serganova* [GGMS]*) Two vertices v^{B_1}, v^{B_2} of P_M form an edge of P_M iff $v^{B_1} - v^{B_2} = e_i - e_j$ for some $i \neq j$, where $e_1, \dots, e_n$ are the standard basis vectors of $\mathbb{R}^n$. In other words, edges of P_M are parallel to roots of $\mathrm{SL}(n, \mathbb{C})$. (In fact, the bases B_1 and B_2 differ by a single exchange iff v^{B_1} and v^{B_2} form an edge of P_M.)*

Conversely, if P is a polytope where all vertices are $0/1$ vectors (each component is either 0 or 1), and each edge of P is parallel to an $\mathrm{SL}(n, \mathbb{C})$ root, then there is a matroid M such that $P = P_M$.

Remark 3.2. *A natural way that matroid polytopes arise is by restricting the momentum mapping $\rho : \mathrm{Gr}_k(\mathbb{C}^n) \to \mathfrak{t}_0^*$ for the action of the maximal compact subtorus T_0 in T on the Grassmannian to the closure of an orbit $T \cdot \Lambda$, see* [GGMS] *or* [BGW]. *The polytope $P_{M(\Lambda)}$ is the image of ρ restricted to the closure of $T \cdot \Lambda$.*

3.2. **Weight polytopes.** Suppose that V is a finite dimensional complex representation of a torus T. Then V is a direct sum of weight spaces,

$$V = \bigoplus_{\mu} V[\mu],$$

where $V[\mu] = \{v \in V \mid t \cdot v = e^{\mu}(t)v \text{ for all } t \in T\}$. Note that a section $s \in V_\lambda = \Gamma(F, L_\lambda)$ is T-invariant for the μ-linearization of L_λ if and only if $s \in V_\lambda[\mu]$.

Given a dominant weight λ let P_λ denote the associated parabolic subgroup. For each $g \in G$, let

$$wt_\lambda(g) = \{\mu \mid (\exists s \in V_\lambda[\mu])(s(gP_\lambda) \neq 0)\}.$$

Let the *weight polytope* $\overline{wt}_\lambda(g)$ be the convex hull of $wt_\lambda(g)$ (the convex hull is taken in $\mathfrak{t}_0^*$, where $\mathfrak{t}_0$ is the Lie algebra of the maximal compact torus $T_0 \subset T$).

Lemma 3.3. *For any two dominant weights λ_1 and λ_2, we have*

$$wt_{\lambda_1}(g) + wt_{\lambda_2}(g) = wt_{\lambda_1+\lambda_2}(g),$$

where the summation denotes the Minkowski sum, $A + B = \{a + b \mid a \in A, b \in B\}$.

Proof. Suppose that $\mu_1 \in wt_{\lambda_1}(g)$ and $\mu_2 \in wt_{\lambda_2}(g)$. Let $s_1 \in V_{\lambda_1}[\mu_1]$ and $s_2 \in V_{\lambda_2}[\mu_2]$ such that $s_1(gP_{\lambda_1}) \neq 0$ and $s_2(gP_{\lambda_2}) \neq 0$. Recall there are functions $f_1, f_2 : G \to \mathbb{C}$ such that $s_1 = s_{f_1}$ and $s_2 = s_{f_2}$. We have that $f_1(g) \neq 0$ and $f_2(g) \neq 0$. Hence, $f_1(g)f_2(g) \neq 0$. The section $s_{f_1 f_2}$ lies in $V_{\lambda_1+\lambda_2}[\mu_1 + \mu_2]$, and is nonzero at $gP_{\lambda_1+\lambda_2}$.

Now suppose that $\mu \in wt_{\lambda_1+\lambda_2}(g)$. We may identify the irreducible representation V_λ as the space of global sections of $\pi^*(L_\lambda)$ of G/B where B is the Borel subgroup of G and $\pi : G/B \to G/P_\lambda$. This is justified since

the pullback $\pi^* : \Gamma(G/P_\lambda, L_\lambda) \to \Gamma(G/B, \pi^*(L_\lambda))$ is an isomorphism of vector spaces. We shall also abuse notation and identify L_λ with the pullback $\pi^* L_\lambda$.

The tensor product $V_{\lambda_1} \otimes V_{\lambda_2}$ is the vector space of sections of the outer tensor product $L_{\lambda_1} \boxtimes L_{\lambda_2}$ of $G/B \times G/B$, where B is the Borel subgroup. The irreducible representation $V_{\lambda_1+\lambda_2}$ is a direct summand of $V_{\lambda_1} \otimes V_{\lambda_2}$, and the projection $V_{\lambda_1} \otimes V_{\lambda_2} \to V_{\lambda_1+\lambda_2}$ is realized by pulling back $L_{\lambda_1} \boxtimes L_{\lambda_2}$ to the diagonal $\Delta \subset G/B \times G/B$. We have assumed there is a section $s \in V_{\lambda_1+\lambda_2}[\mu]$ such that $s(gB) \neq 0$. Clearly $(V_{\lambda_1} \otimes V_{\lambda_2})[\mu]$ surjects onto $V_{\lambda_1+\lambda_2}[\mu]$. Furthermore,

$$(V_{\lambda_1} \otimes V_{\lambda_2})[\mu] = \sum_{\mu_1+\mu_2=\mu} V_{\lambda_1}[\mu_1] \otimes V_{\lambda_2}[\mu_2].$$

Hence there must exist weights μ_1, μ_2 such that $\mu_1 + \mu_2 = \mu$ and some component $s' = s_1 s_2$ of s such that $s_1(gB)s_2(gB) \neq 0$ and $s_1 \in V_{\lambda_1}[\mu_1]$ and $s_2 \in V_{\lambda_2}[\mu_2]$. □

Corollary 3.4. *Suppose that $\lambda = \sum_{i=1}^{n-1} a_i \varpi_i$ is dominant, i.e. each a_i is non-negative and ϖ_i denotes the i-th fundamental weight connected to the Grassmannian $\mathrm{Gr}_i(\mathbb{C}^n)$. Then for any $g \in G$,*

$$wt_\lambda(g) = \sum_{i=1}^{n-1} a_i \cdot wt_{\varpi_i}(g),$$

where the sum indicates Minkowski sum and $a_i \cdot wt_{\varpi_i}(g)$ denotes the a_i-fold Minkowski sum of $wt_{\varpi_i}(g)$.

The weight polytope $\overline{wt}_\lambda(g)$ is a *flag matroid polytope* within the more general setting of Coxeter matroid polytopes, see [BGW]. However, we will only need to consider standard matroid polytopes, as they are the building blocks for flag matroid polytopes.

Proposition 3.5. *Suppose that ϖ_k is the k-th fundamental weight. Then $\overline{wt}_{\varpi_k}(g)$ is a matroid polytope for any $g \in G$.*

Proof. A basis for the sections of L_{ϖ_k} is given by bracket functions $[i_1, i_2, ..., i_k]$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. The section $s = [i_1, i_2, ..., i_k]$ is equal to s_f, where $f : G \to \mathbb{C}$ assigns the determinant of the k by k submatrix given by columns $1, 2, \ldots, k$ and rows $i_1, i_2, ..., i_k$ of $g \in G$. The bracket $[i_1, i_2, ..., i_k]$ belongs to the weight space $V_{\varpi_k}[\mu]$ where $e^\mu = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n/\Delta$ is given by $a_i = 1$ if $i = i_t$ for some t, $1 \leq t \leq k$, otherwise $a_i = 0$. Now suppose that $gP_{\varpi_k} \in G/P_{\varpi_k} = \mathrm{Gr}_k(\mathbb{C}^n)$. The linear subspace defined by gP_{ϖ_k} is the span of the first k columns of g. We have that $\mu \in \overline{wt}_{\varpi_k}(g)$ iff μ is a 0/1 vector (mod Δ) with k ones (occuring at $I = (i_1, i_2, \ldots, i_k)$) and $n - k$ zeros such that the I-th minor is nonzero.

Let $M(g)$ be the matroid with ground set $\{1, 2, \ldots, n\}$ of the vector configuration $r_1, r_2, \ldots, r_n \in \mathbb{C}^k$ where r_i is the i-th row of g restricted to the

first k columns, i.e. $r_i = (g_{i,1}, g_{i,2}, \dots, g_{i,k})$. It is clear that the matroid polytope of $M(g)$ is the weight polytope $\overline{wt_{\varpi_k}}(g)$. $\square$

4. Saturation properties of weight polytopes

We shall prove the following lemma by a combinatorial argument. The main theorem 2.2 easily follows from this lemma. Neil White proved in [W] the exact same statement for $\lambda = \varpi_k$, using a theorem of Edmonds in matroid theory.

Lemma 4.1. *Suppose $g \in G$ and λ is a dominant weight. Suppose μ is a weight such that $\lambda - \mu$ is in the root lattice. Then $N\mu \in wt_{N\lambda}(g)$ implies $\mu \in wt_\lambda(g)$ for all $N > 0$.*

Remark 4.2. *If G is any complex semi-simple group, and λ is a dominant weight, and $\lambda - \mu$ is in the root lattice of G, then $V_{N\lambda}[N\mu] \neq 0$ implies $V_\lambda[\mu] \neq 0$. However, the lemma is a much stronger statement than this (and it does not hold for groups other than $\mathrm{SL}(n, \mathbb{C})$) because g is fixed (i.e., the point $gP_\lambda \in G/P_\lambda$ is fixed).*

Let R be the set of $\mathrm{SL}(n, \mathbb{C})$ roots. Let $Q(R)$ (resp. $P(R)$) denote the root lattice (resp. weight lattice). Convex hulls of subsets of the weight lattice, denoted by an overline, should take place in $\mathfrak{t}_0^*$, which is isomorphic to $P(R) \otimes \mathbb{R} = \overline{P(R)} = \mathbb{R}^n/\Delta$. The map $\epsilon : P(R) \to \mathbb{Z}/n\mathbb{Z}$ given by $\epsilon(a_1, \dots, a_n) = \sum_i a_i \mod n$ is a homomorphism of abelian groups, and $Q(R) = \ker(\epsilon)$.

Definition 4.3. *A finite subset A of $Q(R)$ is called root–saturated if*

- *the convex hull $\overline{A}$ is such that each edge e_i is parallel to a root γ_i in R, (i.e. $\overline{A}$ is a flag matroid polytope, see* [BGW]*.)*
- *for each $x \in A$, $A = \overline{A} \cap Q(R)$.*

We will eventually prove that $wt_\lambda(g) - \lambda$ (the set $wt_\lambda(g)$ translated by $-\lambda$) is root-saturated for any dominant weight λ.

Definition 4.4. *The Minkowski sum of two subsets A,B of Euclidean space, denoted $A + B$, is*

$$A + B = \{a + b \, : \, a \in A, \, b \in B\}$$

Lemma 4.5. *Suppose that $\alpha_1, \dots, \alpha_{n-1} \in R$ are independent over $\mathbb{Q}$. Then they are a basis for the root lattice $Q(R)$.*

Proof. The proof goes by induction on n. If $n = 2$ there are only two roots $\alpha, -\alpha$ and they generate the same lattice. Now suppose that $n > 2$. Let $\mathbb{Z}[\alpha_1, \dots, \alpha_{n-1}]$ be the $\mathbb{Z}$–span of $\alpha_1, \dots, \alpha_{n-1}$. Without loss of generality we may assume that each α_i is a positive root since negating α_i does not change the span over $\mathbb{Z}$. Let $\sigma_1, \dots, \sigma_{n-1}$ be the standard simple roots of $SL(n)$. That is, $\sigma_i = e_i - e_{i+1}$. Note that any positive root $e_i - e_j = \sum_{t=i}^{j-1} \sigma_t$ is a sum of consecutive simple roots. Conversely any consecutive sum of simple

roots is a positive root. We may choose some $w \in W$ (where W is the Weyl group) such that $w(\alpha_{n-1}) = \sigma_{n-1}$. In particular if $\alpha_{n-1} = e_i - e_j$ let w be the product of two cycles $(n-1 \quad i)(n \quad j)$. Since elements of W induce isomorphisms of the lattice $Q(R)$, we have that $w(\alpha_1), \ldots, w(\alpha_{n-1})$ is a basis of $Q(R)$ if and only if $\alpha_1, \ldots, \alpha_{n-1}$ is a basis of $Q(R)$. Reassign $\alpha_i := w(\alpha_i)$. For each $i \leq n-2$, if $\alpha_i = e_s - e_n = \sigma_s + \cdots + \sigma_{n-1}$ replace α_i with $\alpha_i - \sigma_{n-1} = \sigma_s + \cdots + \sigma_{n-2} = e_s - e_{n-1}$. Now the roots $\alpha_1, \ldots, \alpha_{n-2}$ may be identified with roots of $SL(n-1)$. By the induction hypothesis $\mathbb{Z}[\alpha_1, \ldots, \alpha_{n-2}] = \mathbb{Z}[\sigma_1, \ldots, \sigma_{n-2}]$. Since $\alpha_{n-1} = \sigma_{n-1}$ we have that $\mathbb{Z}[\alpha_1, \ldots, \alpha_{n-1}] = Q(R)$.

□

Lemma 4.6. *Suppose that A and B are root-saturated, and $\overline{A} \cap \overline{B}$ is nonempty. Then $A \cap B$ is nonempty.*

Proof. The proof is by induction on the dimension of $\overline{A}$. If $\dim \overline{A} = 0$ then $A = \{a\}$ for some $a \in Q(R)$. Then $\overline{A} \cap \overline{B} = A \cap B = \{a\}$. Now suppose that $\dim \overline{A} \geq 1$.

We have two cases, the first case is that the intersection $\overline{A} \cap \overline{B}$ contains a boundary point of $\overline{A}$. Then there is some facet F of $\overline{A}$ such that $F \cap \overline{B}$ is nonempty. We claim $F \cap A$ is root–saturated. The vertices of F are within $F \cap A$. so $\overline{F \cap A} \supset F$. On the other hand $F \subset \overline{F \cap A}$ so $F \subset \overline{F \cap A}$; therefore $F = \overline{F \cap A}$. The edges of F are also edges of $\overline{A}$ hence they are parallel to roots. Furthermore, for any $x \in F \cap A$, we have $\overline{F \cap A} \cap Q(R) \subset A$ since A is root–saturated, and it follows that $\overline{F \cap A} \cap Q(R) = F \cap A$ since $F \cap A \subset A \subset Q(R)$. Since $\dim F < \dim \overline{A}$ we may apply the induction hypothesis to get that $F \cap A \cap B$ is nonempty and hence $A \cap B$ is nonempty.

On the other hand suppose that $\overline{A} \cap \overline{B}$ contains no boundary point of $\overline{A}$. Let $L_A(R)$ be the sub–lattice of $Q(R)$ spanned by the roots which are parallel to some edge of $\overline{A}$. Let $a_0 \in A$ be a vertex of $\overline{A}$. Note that the affine space $H_A = a_0 + \overline{L_A(R)}$ is the smallest affine space containing $\overline{A}$. We claim $H_A \cap \overline{B} = \overline{A} \cap \overline{B}$. Suppose that $z \in H_A \cap \overline{B}$. Let $a \in \overline{A} \cap \overline{B}$. Since H_A has the same dimension as $\overline{A}$, there are linear inequalities $\eta_i(x) \leq f_i$ where the interior of $\overline{A}$ consists of points $x \in H_A$ where the inequalities are strict; that is, $\eta_i(x) < f_i$ for all i if and only if x is an interior point of $\overline{A}$. The boundary points of $\overline{A}$ are those points $x \in \overline{A}$ such that $\eta_i(x) = f_i$ for some i. Let $c(t) = (1-t)a + tz$ for $0 \leq t \leq 1$. Suppose that $z \notin \overline{A}$. Then there is some i such that $\eta_i(z) > f_i$. However a is an interior point of $\overline{A}$ and so $\eta_i(a) < f_i$. Hence there is some t_0 such that $\eta(c(t_0)) = f_i$ in which case $c(t_0)$ is a boundary point of $\overline{A}$. But $c(t) \in \overline{B}$ for each t by convexity of $\overline{B}$. This contradicts that $\overline{A} \cap \overline{B}$ is disjoint from the boundary of A. Hence $H_A \cap \overline{B} = \overline{A} \cap \overline{B}$. Therefore $(H_A \cap Q(R)) \cap B = A \cap B$ since $\overline{A} \cap Q(R) = A$ and $\overline{B} \cap Q(R) = B$.

We now show by induction on $\dim \overline{B}$, that for any B which is root-saturated, that $H_A \cap \overline{B}$ is nonempty implies $(H_A \cap Q(R)) \cap B$ is nonempty.

Suppose that $\dim \overline{B} = 0$. Then $B = \{b\}$ for some $b \in Q(R)$, and so $b \in (H_A \cap Q(R)) \cap B$. Now suppose that $\dim \overline{B} \geq 1$. We have two cases.

First suppose that $H_A \cap \overline{B}$ intersects the boundary of $\overline{B}$ nontrivially. Then there is a face F of $\overline{B}$ such that $H_A \cap F$ is nonempty. Since $F \cap B$ is root–saturated, $\overline{F \cap B} = F$, $H_A \cap F$ is nonempty, and $\dim F < \dim B$, we may apply the induction hypothesis and we're finished.

Now suppose that $H_A \cap \overline{B}$ is disjoint from the boundary of $\overline{B}$. Let $L_B(R)$ be the sub–lattice of $Q(R)$ spanned by the roots which are parallel to some edge of $\overline{B}$. Let $b_0 \in B$ be a vertex of $\overline{B}$. The affine space $H_B = b_0 + \overline{L_B(R)}$ is the smallest affine space containing $\overline{B}$. As above, we have that $H_A \cap H_B = H_A \cap \overline{B}$ and so $(H_A \cap Q(R)) \cap (H_B \cap Q(R)) = A \cap B$. Since H_A does not intersect the boundary of $\overline{B}$, we have that $H_A \cap H_B$ is a single point z_0, since if the dimension of the intersection $H_A \cap H_B = H_A \cap \overline{B}$ is greater than zero then $H_A \cap \overline{B}$ is unbounded. But $\overline{B}$ is compact since B is finite and this cannot happen. We now show that $z_0 \in Q(R)$. We have that $z_0 = a_0 + v_A = b_0 + v_B$ where $a_0 \in A$, $b_0 \in B$, $v_A \in \overline{L_A(R)}$, $v_B \in \overline{L_B(R)}$. Let $\{\alpha_1, \dots, \alpha_p\} \subset R$ be a basis of $L_A(R)$ and let $\{\beta_1, \dots, \beta_q\} \subset R$ be a basis of $L_B(R)$. Since the intersection of H_A and H_B is a point, we have that $\overline{L_A(R)} \cap \overline{L_B(R)} = \{0\}$. Hence the set $\{\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q\}$ is linearly independent in $\overline{Q(R)}$. Choose $\{\gamma_1, \dots, \gamma_r\} \subset R$ so that $\{\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q, \gamma_1, \dots, \gamma_r\}$ is a basis for $\overline{Q(R)}$. By the Lemma above this is also a basis for the lattice $Q(R)$. Now $v_A = \sum_i c_i \alpha_i$ and $v_B = \sum_j d_j \beta_j$ are unique expressions for v_A, v_B. But also the difference $a_0 - b_0 = v_B - v_A = (\sum_j d_j \beta_j) - (\sum_i c_i \alpha_i)$ lies within the lattice $Q(R)$, and so the coefficients c_i,d_j must be integers. Hence, z_0 is a lattice point and we've finished the proof of the Lemma. □

Theorem 4.7. *Suppose that A and B are root-saturated. Then the Minkowski sum $A + B$ is root-saturated.*

Proof. We show that the Minkowski sum $A+B$ is root-saturated if A and B are each root-saturated. Clearly $A+B$ is finite, and the elements are within $Q(R)$ since $Q(R)$ is closed under addition. First we show that the edges of $\overline{A+B}$ are parallel to roots. Clearly $\overline{A} + \overline{B} = \overline{A+B}$. The Minkowski sum of two polytopes P, Q has edges of the following types:

- (vertex of P) + (edge of Q).
- (edge of P) + (vertex of Q).
- (edge of P) + (edge of Q), providing these edges are parallel.

We leave the proof to the reader (the proof is easily obtained by observing that the fan of $P + Q$ is the meet of the fan of P with the fan of Q). In all three cases, the resulting edge is parallel to an edge of either P or Q or both, and hence it is parallel to some root in R.

Next we must show that $A + B = (\overline{A+B}) \cap Q(R)$. Suppose that $z \in (\overline{A+B}) \cap Q(R)$. Hence there exists $x \in \overline{A}$ and $y \in \overline{B}$ such that $x + y = z$. Hence $x \in (z + \overline{-B}) \cap \overline{A}$, where $-B = \{-b : b \in B\}$. Clearly $z + (-B)$ is root-saturated. Hence, we may apply the Lemma above to get a lattice

point x_0 in the intersection. Since A is saturated, we have that $x_0 \in A$. Now we have that $z = x_0 + y_0$ where $y_0 \in \overline{B}$. But since $z, x_0 \in Q(R)$ we have that $y_0 = z - x_0 \in Q(R)$, and so $y_0 \in B$ since B is root-saturated, and we're finished. □

Lemma 4.8. *If ϖ_k is a fundamental weight and $g \in G$ then the translation $wt_{\varpi_k}(g) - \varpi_k$ is root-saturated.*

Proof. Note that all elements of $wt_{\varpi_k}(g)$ are 0/1 vectors (mod Δ) having k ones and $n-k$ zeros. Translating by $-\varpi_k$ results in vectors whose first k components may be either 0 or -1 and last $n-k$ components are 0 or $+1$, and the sum of all components is zero. Hence the first k components define a vertex of the negated unit k-cube, and the last $n-k$ components are vertices of the $n-k$-cube. Therefore, there can be no additional lattice points in the convex hull. We already showed that the convex hull of $wt_{\varpi_k}(g)$ is a matroid polytope, so the edges are parallel to roots. This property is preserved by translations. □

Corollary 4.9. *For any dominant weight λ and $g \in G$, the set $wt_\lambda(g) - \lambda$ is root-saturated.*

Proof. We have that $\lambda = \sum_{k=1}^{n-1} a_k \varpi_k$, where the a_k's are non-negative integers. Also, $wt_\lambda(g) = \sum_{k=1}^{n-1} a_k \cdot wt_{\varpi_k}(g)$ (Minkowski sum). Hence,

$$wt_\lambda(g) - \lambda = \sum_{k=1}^{n-1} a_k \cdot (wt_{\varpi_k}(g) - \varpi_k).$$

Since the root-saturated property is preserved under Minkowski sums, we have that $wt_\lambda(g) - \lambda$ is root-saturated. □

Proof of lemma 4.1.

Proof. Suppose that $N\mu \in wt_{N\lambda}(g)$. Then $N(\mu - \lambda) \in wt_{N\lambda}(g) - N\lambda$. The convex hull of $wt_{N\lambda}(g) - N\lambda$ scaled by $1/N$ is equal to the convex hull of $wt_\lambda(g) - \lambda$ since $N \cdot wt_\lambda(g) = wt_{N\lambda}(g)$. Therefore $\mu - \lambda$ is in the convex hull of $wt_\lambda(g) - \lambda$. But since $\mu - \lambda \in Q(R)$ and $wt_\lambda(g) - \lambda$ is root-saturated, we have that $\mu - \lambda \in wt_\lambda(g) - \lambda$, so $\mu \in wt_\lambda(g)$. □

Proof of main theorem 2.2.

Proof. Suppose that gP_λ is semistable relative to the μ-linearization of the line bundle L_λ. This means there is some $N > 0$ and a section $s \in \Gamma(G/P_\lambda, L_\lambda^{\otimes N})^T$ such that $s(gP_\lambda) \neq 0$. This means that $N\mu \in wt_{N\lambda}(g)$. By Lemma 4.1 we have that $\mu \in wt_\lambda(g)$. So there must exist a section $s' \in \Gamma(G/P_\lambda, L_\lambda)^T$ such that $s'(gP_\lambda) \neq 0$. □

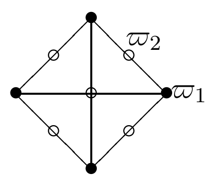
4.1. **Failure of main theorem for $G = \mathrm{SO}(5, \mathbb{C})$.** Let $B(z, w)$ be the bilinear form on $\mathbb{C}^5$ given by

$$B(z, w) = z_1 w_5 + z_2 w_4 + z_3 w_3 + z_4 w_2 + z_5 w_1 = 2z_1 w_5 + 2z_2 w_4 + z_3 w_3.$$

Now $\mathrm{SO}(5, \mathbb{C}) \subset \mathrm{SL}(5, \mathbb{C})$ is the subgroup preserving B. The maximal torus T may be taken to the diagonal matrices in $\mathrm{SO}(5, \mathbb{C})$. Elements of T have the form $diag(t_1, t_2, 1, 1/t_2, 1/t_1)$ for $t_1, t_2 \in \mathbb{C}^*$. Let ϖ_1 denote the first fundamental weight of $\mathrm{SO}(5, \mathbb{C})$. We have $e^{\varpi_1}(t_1, t_2, 1, 1/t_1, 1/t_2) = t_1$, but the second fundamental weight does not lift to a character of $\mathrm{SO}(5, \mathbb{C})$ - one needs to go the universal cover to find such a character. Let $P_{\varpi_1} \subset \mathrm{SO}(5, \mathbb{C})$ be the associated parabolic subgroup. The quotient space $\mathrm{SO}(5, \mathbb{C})/P_{\varpi_1}$ may be identified with the space of isotropic lines in $\mathbb{C}^5$.

Let x be the (isotropic) line through $(1, \sqrt{-1}, 0, \sqrt{-1}, 1)$. Let $g_x \in \mathrm{SO}(5, \mathbb{C})$ be such that $g_x P_{\varpi_1} = x$. The set $wt_{\varpi_1}(g_x)$ is equal to $\{\varpi_1, 2\varpi_2 - \varpi_1, -2\varpi_2 + \varpi_1, -\varpi_1\}$. Depiction:

$wt_{\varpi_1}(g_x) =$



This set is missing the origin, although $V_{\varpi_1}[0] \neq 0$ and $\varpi_1 \in Q(\mathrm{SO}(5, \mathbb{C}))$, so $wt_{\varpi_1}(g_x) - \varpi_1$ is not root-saturated. Note the origin does belong to $wt_{2\varpi_1}(g_x) = wt_{\varpi_1}(g_x) + wt_{\varpi_1}(g_x)$. Therefore x is semistable for the democratic linearization of L_{ϖ_1}. It follows that for the democratic linearization of L_{ϖ_1}, one requires a T-invariant section of $L_{\varpi_1}^{\otimes 2}$ to pick out the semistable point x.

5. Projective Normality

Let H be the group of diagonal matrices in $\mathrm{GL}(n, \mathbb{C})$. Hence $T \subset H$ is the set of diagonal matrices with determinant one. Let $\chi_1, \ldots, \chi_m$ be m characters of H. That is, each $\chi_i : H \to \mathbb{C}^*$ is an algebraic homomorphism of groups. Each χ_i is given by a point $\mathbf{a}_i = (a_{i,1}, \ldots, a_{i,n}) \in \mathbb{Z}^n$, where

$$\chi_i(h_1, \ldots, h_n) = \prod_{j=1}^{n} h_j^{a_{i,j}}.$$

These characters determine an action of H on $\mathbb{A}^m$ by

$$h \cdot (z_1, z_2, \ldots, z_m) = (\chi_1(h) z_1, \chi_2(h) z_2, \ldots, \chi_m(h) z_m).$$

Now take any point $z \in \mathbb{A}^m$, and let $X(z)$ be the Zariski closure of the H-orbit of z. That is, $X(z) = cl(H \cdot z)$. Certainly $X(z)$ contains a dense torus and there is a natural action of this torus on $X(z)$; so $X(z)$ is a (possibly non-normal) toric variety.

But when is $X(z)$ a *normal* toric variety, i.e. when is the coordinate ring of $X(z)$ integrally closed in its field of fractions? Some notation: if A is a

finite subset of $\mathbb{Z}^d$ then let $\mathbb{Z}(A)$ be the sub-lattice generated by A, let $\mathbb{N}(A)$ be the semigroup of all non-negative integral combinations of elements of A, and let $\mathbb{Q}_0^+(A)$ be the rational cone in $\mathbb{Q}^d$ given by all non-negative rational combinations of elements of A. According to Proposition 13.5 of [St] we have that the semigroup algebra $\mathbb{C}[\mathbb{N}(A)]$ is normal iff $\mathbb{N}(A) = \mathbb{Z}(A) \cap \mathbb{Q}_0^+(A)$.

The following proposition is likely well known but we give a proof for lack of reference.

Proposition 5.1. *Let* $\mathrm{supp}(z) = \{i \mid z_i \neq 0\}$. *Let* $A(z) = \{\chi_i \mid i \in \mathrm{supp}(z)\}$. *Then $X(z)$ is isomorphic to the affine toric variety defined by $A(z) \subset \mathbb{Z}^n$. That is, $X(z)$ is isomorphic to the affine variety $V \subset \mathbb{C}^{\#A(n)}$ of the semigroup algebra $\mathbb{C}[\mathbb{N}(A(z))]$, where $\mathbb{N}(A(z))$ is the semigroup in $\mathbb{Z}^n$ generated by $A(z)$. Hence $X(z)$ is normal if and only if $\mathbb{Z}(A(z)) \cap \mathbb{Q}_0^+(A(z)) = \mathbb{N}(A(z))$.*

Proof. Let $\bar{z} \in \mathbb{C}^m$ be given by $\bar{z}_i = 1$ if $i \in \mathrm{supp}(z)$ and $\bar{z}_i = 0$ otherwise. Let $s_i = 1/z_i$ if $z_i \neq 0$ and $s_i = 1$ if $z_i = 0$. Then the matrix $diag(s_1, \ldots, s_m)$ defines an algebraic automorphism of $\mathbb{A}^m$ which takes $X(z)$ to $X(\bar{z})$, so $X(\bar{z})$ is isomorphic to $X(z)$. Hence we may assume that all components of z are either 0 or 1. Additionally, $X(z)$ lives entirely within the components i where z_i is nonzero. Hence, we may project $X(z)$ onto the linear subspace given by the components in $\mathrm{supp}(z)$, which defines an isomorphism of $X(z)$ onto its image. Thus, we may assume that each component of z is equal to one. If $\chi_i = \chi_j$ for some i, j, we may also project away one of these. Hence we have reduced to the case that the χ_i's are distinct, and z is the vector of all ones. The coordinate ring of $X(z)$ is now easily seen to be the semigroup algebra $\mathbb{C}[\mathbb{N}(A(z))]$. $\square$

A dominant weight λ of $\mathrm{SL}(n, \mathbb{C})$ may be lifted to a dominant weight $\widetilde{\lambda}$ of $\mathrm{GL}(n, \mathbb{C})$ by normalizing λ so that the last component is zero. That is, the image of $\widetilde{\lambda} \in \mathbb{Z}^n$ in $\mathbb{Z}^n/\Delta$ is λ, and $\widetilde{\lambda}_n = 0$. Let

$$|\widetilde{\lambda}| = \sum_{i=1}^{n} \widetilde{\lambda}_i.$$

Now V_λ is also an irreducible representation of $\mathrm{GL}(n, \mathbb{C})$, where $zI_n \in \mathrm{GL}(n, \mathbb{C})$ acts by scaling each vector $s \in V_\lambda$ by $z^{|\widetilde{\lambda}|}$, and so if $\widetilde{g} = zg$ where $z \in \mathbb{C}^*$ and $g \in \mathrm{SL}(n, \mathbb{C})$ then the action of $\widetilde{g}$ is defined by $\widetilde{g} \cdot s = z^{|\widetilde{\lambda}|}(g \cdot s)$. A basis for the representation V_λ is given by semi-standard tableaux τ of shape $\widetilde{\lambda}$ (with total number of slots equal to $|\widetilde{\lambda}|$), filled with indices from 1 to n. A section $s_\tau \in V_\lambda[\mu]$ iff the number of times the index i appears in τ is equal to μ_i. Here we are treating μ as a weight of $\mathrm{GL}(n, \mathbb{C})$. Note that if $V_\lambda[\mu] \neq 0$ then $|\mu| = \sum_{i=1}^n \mu_i = |\widetilde{\lambda}|$ since $|\mu|$ must equal the total number of slots in τ, where $s_\tau \in V_\lambda[\mu]$.

Recall that $H = \mathbb{C}^*(T)$ is the maximal torus in $\mathrm{GL}(n,\mathbb{C})$ consisting of diagonal matrices. For each $g \in \mathrm{GL}(n,\mathbb{C})$ let

$$wt_{\widetilde{\lambda}}(g) = \{\mu \mid (\exists s \in V_\lambda[\mu])(s(gP_{\widetilde{\lambda}}) \neq 0)\},$$

where $P_{\widetilde{\lambda}} \subset \mathrm{GL}(n,\mathbb{C})$ is the parabolic subgroup $\mathbb{C}^*(P_\lambda)$ associated to $\widetilde{\lambda}$. Each $\mu \in wt_{\widetilde{\lambda}}(g) \subset \mathbb{Z}^n$ satisfies $|\mu| = |\widetilde{\lambda}|$.

Note that the root lattice of $\mathrm{SL}(n,\mathbb{C})$ may be identified with integral vectors $v \in \mathbb{Z}^n$ whose components sum to zero. Hence, for any $g \in \mathrm{SL}(n,\mathbb{C})$ we have an identification of $wt_\lambda(g) - \lambda$ with $wt_{\widetilde{\lambda}}(g) - \widetilde{\lambda}$. In particular, $wt_{\widetilde{\lambda}}(g) - \widetilde{\lambda}$ is root-saturated.

Let $N_{\widetilde{\lambda}}$ be the sub-lattice of $\mathbb{Z}^n$ given by

$$N_{\widetilde{\lambda}} = \{v \in \mathbb{Z}^n \mid \ |v| = \sum_{i=1}^{n} v_i \equiv 0 \mod |\widetilde{\lambda}|\}.$$

Lemma 5.2. *For any $g \in \mathrm{SL}(n,\mathbb{C})$,*

$$\mathbb{Q}_0^+(wt_{\widetilde{\lambda}}(g)) \cap N_{\widetilde{\lambda}} = \mathbb{N}(wt_{\widetilde{\lambda}}(g)).$$

Proof. Suppose that $v \in \mathbb{Q}_0^+(wt_\lambda(g)) \cap N_{\widetilde{\lambda}}$. Then $|v| = d|\widetilde{\lambda}|$ for some $d \in \mathbb{N}$. Hence v belongs to the convex hull of the d-th dilate of $wt_{\widetilde{\lambda}}(g)$, so v is in the convex hull of $wt_{d\widetilde{\lambda}}(g)$, since $wt_{d\widetilde{\lambda}}(g)$ is the d-fold Minkowski sum of $wt_{\widetilde{\lambda}}(g)$. But $wt_{d\widetilde{\lambda}}(g) - d\widetilde{\lambda}$ is root-saturated, and since $v - d\widetilde{\lambda} \in Q(R)$ we have that $v - d\widetilde{\lambda} \in wt_{d\widetilde{\lambda}}(g) - d\widetilde{\lambda}$. Equivalently, $v \in wt_{d\widetilde{\lambda}}(g)$. Since $wt_{d\widetilde{\lambda}}(g)$ is the d-fold Minkowski sum of $wt_{\widetilde{\lambda}}(g)$, we have that $v \in \mathbb{N}(wt_{\widetilde{\lambda}}(g))$. $\square$

Corollary 5.3. *The semigroup algebra $\mathbb{C}[\mathbb{N}(wt_{\widetilde{\lambda}}(g))]$ is normal.*

Now suppose that λ is dominant and P_λ is the associated parabolic subgroup. Choose a basis $(s_1, s_2, \ldots, s_N)$ of $V_\lambda = \Gamma(\mathrm{SL}(n,\mathbb{C})/P_\lambda, L_\lambda)$ such that each basis vector is a generalized eigenvector for the democratic T-action. (Recall the democratic action is the restriction of the natural action of $\mathrm{SL}(n,\mathbb{C})$ on V_λ to T.) Let $\iota_\lambda : \mathrm{SL}(n,\mathbb{C})/P_\lambda \to \mathbb{P}(V_\lambda)$ be the projective embedding determined by this choice of basis. (Note that one typically embeds G/P_λ into $\mathbb{P}(V_\lambda^*)$ as there is no need for a choice of basis, but it is more convenient for us to embed into $\mathbb{P}(V_\lambda)$.)

The following theorem has been proven by R. Dabrowski for certain *generic* T-orbits in G/P for G an arbitrary semi-simple complex Lie group, see [Dab]. Herein lies the first proof for *arbitrary* T-orbits in the case $G = \mathrm{SL}(n,\mathbb{C})$.

Theorem 5.4. *The Zariski closure of any T-orbit in $\mathrm{SL}(n,\mathbb{C})/P_\lambda \hookrightarrow \mathbb{P}(V_\lambda)$ is a projectively normal toric variety.*

Proof. Let $x \in \mathrm{SL}(n,\mathbb{C})/P_\lambda \subset \mathbb{P}(V_\lambda)$. Let $cl(T \cdot x)$ denote the Zariski closure of the orbit $T \cdot x$. Let $\mathrm{Aff}(cl(T \cdot x)) \subset V_\lambda$ denote the associated affine cone; it is easy to see that $\mathrm{Aff}(cl(T \cdot x)) = cl(H \cdot v_x)$ where v_x is any nonzero vector

on the line x, since the scalar matrices in H fill out all nonzero multiples of points in $T \cdot v_x$.

Let $g \in \mathrm{SL}(n, C)$ be such that $gP_\lambda = x$. Now $wt_{\tilde{\lambda}}(g) = A(v_x)$. Hence by Proposition 5.1, the affine toric variety $\mathrm{Aff}(cl(T \cdot x))$ is normal if and only if the semigroup algebra $\mathbb{C}[\mathbb{N}(wt_{\tilde{\lambda}}(g))]$ is normal, which we have already shown. This means that the projective toric variety $cl(T \cdot x)$ is projectively normal. □

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